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Constructing an autonomous system with infinitely many chaotic attractors

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Some classical chaotic systems such as the Lorenz system and Chua system have finite numbers of chaotic attractors. This letter develops a simple, effective method for constructing lower-dimensional autonomous systems with infinitely many chaotic attractors. As an application, a Lorenz-type system and a Rössler-type system with infinitely many chaotic attractors are constructed with bifurcation analysis, and with an extension to the fractional-order setting. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4986356]

This letter suggests a general method for the construction of lower-dimensional autonomous systems with infinitely many chaotic attractors, and with an extension to the fractional-order setting. Some new phenomena generalize the classical concept of sensitive dependence on the initial conditions, that is, the system orbits beginning from two very close initial points might either converge and then stay on one attractor or diverge to two different attractors. This method also relates to the well-known second part of Hilbert's sixteenth problem. Namely, the number of attractors might be determined by the degrees of the polynomial functions in the model.

I. INTRODUCTION

Consider an autonomous ordinary differential equation, $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ and $\Phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. The equilibria of this system are real solutions to $\Phi(\mathbf{x}) = 0$, if they exist. The equilibrium is an attractor if it is stable, and the system may have other attractors such as limit cycles and chaotic attractors. Furthermore, if the basin of attraction for an attractor intersects with a small neighborhood of equilibrium, then the attractor is said to be self-excited; otherwise, it is hidden.[13,14] Hidden attractors can be found in, e.g., the Chua system.[11] Also importantly, the types of equilibria affect the geometric structures of such attractors, and the number and the stability of equilibria also affect the global dynamics of the underlying systems.[24]

It is well known that nonautonomous systems might have many attractors, including chaotic and strange non-chaotic attractors,[25,26,28] but all known autonomous systems have only one chaotic attractor, like the Lorenz system and Chen system,[2,18] or a finite number of attractors, like the Chua system, including self-excited and hidden attractors. The property of the complicated dynamical systems with many coexisting attractors is referred to as the multistability.[3,8,21] There is a controversial theory in astronomy, the "parallel universe theory," claiming that our universe is not the only one but many universes coexist in parallel, which might be explained by quantum mechanics, string theory, or others.[7,23] One may wonder if the chaotic universe model with multistability having multi-attractors could provide yet another refection or explanation of such super-advanced theories. These observations greatly motivate the present research on autonomous systems with many chaotic attractors, which are deemed meaningful and ultimately useful.

This letter aims to develop a simple and effective method for constructing autonomous systems (Lorenz-type, Rössler-type, Chua-type, and so on) with infinitely many attractors (including self-excited and hidden attractors).

First, some new phenomena from classical chaotic systems are found, which expand the notion of the classical concept of sensitive dependence on initial conditions, that is, the system orbits starting from two very close initial points might either converge and then stay on one attractor or diverge to two different attractors.

Second, it is found that a system with many chaotic attractors could be used as a model for the coexistence of many small "universes" (attractors), and these "universes" are separated from each other by the boundaries of their corresponding basins of attraction. These models might be thought of as "parallel chaotic systems" (with multiple attractors).

Third, the new method is also related to the well-known second part of Hilbert's sixteenth problem, where the number of limit cycles of a planar polynomial system is determined by the degrees of the polynomial vector fields. It turns out that the number of attractors in the considered chaotic system is also determined and can be estimated by the degrees of the polynomial functions used in the model and the method. A chaotic analog of the second part of Hilbert's sixteenth problem for autonomous multi-dimensional polynomial systems was formulated in Ref. 12, and a similar problem for nonautonomous systems was provided in Ref. 28.

Finally, the method could be extended to the fractional-order setting, to construct corresponding autonomous systems with infinitely many chaotic attractors.

II. THE MAIN METHOD

Now, precisely, assume that $\phi(t, t_0, w_0)$ is a solution of the initial value problem

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$$\begin{cases} \dot{\mathbf{w}} = F(\mathbf{w}) \\ \mathbf{w}(t_0) = w_0, \end{cases} \tag{2.1}$$

where $t_0 \in \mathbb{R}$ is the initial time and $w_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ is the initial condition. Let Ω be a compact subset of $\mathbb{R}^3$, and $d(w, A) := \inf_{w' \in A} ||w - w'||$ be the distance between the point w and the set A, where $||\cdot||$ is the Euclidean distance on $\mathbb{R}^3$. Set $\Omega_\epsilon := \{w : d(w, \Omega) < \epsilon\}$.

Then, consider the initial value problem (2.1). On the compact set $\Omega \subset \mathbb{R}^3$, if

$$\lim_{t\to\infty} d\big(\phi(t, t_0, w_0), \Omega\big) = 0, \ \forall w_0 \in \mathbb{R}^3 \backslash \Omega,$$

that is, if for any $\epsilon > 0$, there exists $T > t_0$ such that $\phi(t, t_0, w_0) \in \Omega_\epsilon$ for any $t \geq T$, then the set Ω is called an ultimate bound for the system (2.1). If $\phi(t, t_0, w_0) \in \Omega$ for any $w_0 \in \Omega$ and any $t \geq t_0$, then Ω is said to be a positive invariant set for the system (2.1) [Ref. 16, Definition 1].

If a positive invariant set for an autonomous system is contained in a bounded set, then the basin of attraction of the corresponding attractor is also contained in the bounded set. This property is useful in chaos control, chaos synchronization, and many related studies. For instance, the estimate of the boundedness of a positive invariant set for the Lorenz system is an important topic of research.[10,17]

Next, a new method for constructing an autonomous system with infinitely many chaotic attractors is introduced, using three-dimensional models for illustration.

Consider an autonomous system $\dot{\mathbf{p}} = F(\mathbf{p})$, with an attractor having its basin of attraction being contained in the set $\{(p_1, p_2, p_3) : \max_{1\leq i\leq 3}|p_i| \leq C\}$, where constant $C > 0$. By using the coordinate transformation

$$\begin{cases} p_1 = R_1 u_1 \\ p_2 = R_2 u_2 \\ p_3 = R_3 u_3, \end{cases}$$

a system $\dot{\mathbf{u}} = \Phi(\mathbf{u})$ is obtained, where $R_i > 0$ are constant parameters that should be taken large enough, and the basin of attraction is contained in the set $\{(u_1, u_2, u_3) : |u_i| \leq C/R_i, i = 1, 2, 3\}$.

Consider three continuous functions $\phi_i(x_i)$, $i = 1, 2, 3$, with infinitely many simple zeros. By properly choosing parameters R_i, one can obtain the following systems possibly with infinitely many chaotic attractors: (i) $\dot{\mathbf{x}} = \Phi\big(\phi_1(x_1), x_2, x_3\big)$ with the basin of attraction located along the x_1-axis and contained in $\{(x_1, x_2, x_3) : |x_2| \leq C/R_2, |x_3| \leq C/R_3\}$; (ii) $\dot{\mathbf{x}} = \Phi\big(x_1, \phi_2(x_2), x_3\big)$ with the basin of attraction located along the x_2-axis, belonging to $\{(x_1, x_2, x_3) : |x_1| \leq C/R_1, |x_3| \leq C/R_3\}$; (iii) $\dot{\mathbf{x}} = \Phi\big(x_1, x_2, \phi_3(x_3)\big)$ with the basin of attraction located along the x_3-axis and contained in $\{(x_1, x_2, x_3) : |x_1| \leq C/R_1, |x_2| \leq C/R_2\}$; (iv) $\dot{\mathbf{x}} = \Phi\big(\phi_1(x_1), \phi_2(x_2), \phi_3(x_3)\big)$ with attractors "almost everywhere." Here, it is noted that for general continuous functions ϕ_i, the number of the attractors for the system $\dot{\mathbf{x}} = \Phi\big(\phi_1(x_1), \phi_2(x_2), \phi_3(x_3)\big)$ is related to the number of zeros of ϕ_i.

III. CONTINUOUS SYSTEMS WITH INFINITELY MANY CHAOTIC ATTRACTORS

In the following, the above-described method is applied to construct a Lorenz-type system with infinitely many chaotic attractors.

Start with the classical Lorenz system, which has only a chaotic attractor

$$\begin{cases} \dot{p}_1 = 10(p_2 - p_1) \\ \dot{p}_2 = 28p_1 - p_1 p_3 - p_2 \\ \dot{p}_3 = p_1 p_2 - \dfrac{8}{3} p_3. \end{cases}$$

By Ref. 16 [Theorem 1], the positive invariant set is contained in the set $p_1^2 + p_2^2 + (p_3 - 38)^2 < 1600$. Consequently, the attractor is contained in the region

$$\{(p_1, p_2, p_3) : |p_1| < 40, \quad |p_2| < 40, \quad -2 < p_3 < 80\}.$$

Using the following coordinate transformation:

$$\begin{cases} p_1 = R_0 u_1 \\ p_2 = R_0 u_2 \\ p_3 = 2R_0 u_3, \end{cases}$$

one obtains

$$\begin{cases} \dot{u}_1 = 10(u_2 - u_1) \\ \dot{u}_2 = 28u_1 - 2R_0 u_1 u_3 - u_2 \\ \dot{u}_3 = \dfrac{R_0}{2} u_1 u_2 - \dfrac{8}{3} u_3, \end{cases}$$

which will be denoted by $\dot{\mathbf{u}} = \Phi(\mathbf{u})$.

So, the basin of attraction is contained in the set $\{(u_1, u_2, u_3) : \max_{1\leq i\leq 3}|u_i| \leq 40/R_0\}$. Let $\phi_i(x_i) = \sin(x_i)$, $1 \leq i \leq 3$. Then, the system $\dot{\mathbf{x}} = \Phi\big(\sin(x_1), x_2, x_3\big)$ might have infinitely many chaotic attractors along the x_1-axis; the system $\dot{\mathbf{x}} = \Phi\big(x_1, \sin(x_2), x_3\big)$ might have infinitely many chaotic attractors along the x_2-axis; the system $\dot{\mathbf{x}} = \Phi\big(x_1, x_2, \sin(x_3)\big)$ might have infinitely many chaotic attractors along the x_3-axis. As a result, the system $\dot{\mathbf{x}} = \Phi\big(\sin(x_1), \sin(x_2), \sin(x_3)\big)$

$$\begin{cases} \dot{x}_1 = 10\big(\sin(x_2) - \sin(x_1)\big) \\ \dot{x}_2 = 28\sin(x_1) - 2R_0 \sin(x_1)\sin(x_3) - \sin(x_2) \\ \dot{x}_3 = \dfrac{R_0}{2}\sin(x_1)\sin(x_2) - \dfrac{8}{3}\sin(x_3) \end{cases} \tag{3.1}$$

might have infinitely many chaotic attractors all over the three-dimensional space.

Bifurcation analysis has been performed on system (3.1), with bifurcating parameter R_0 which takes only positive integer values for simplicity. The numerical experiments indicate that there is no attractor for $R_0 < 17$; the coexistence of chaotic attractors and periodic orbits is observable for $17 \leq R_0 \leq 18$; the coexistence of many chaotic attractors is observable for $R_0 \geq 19$.

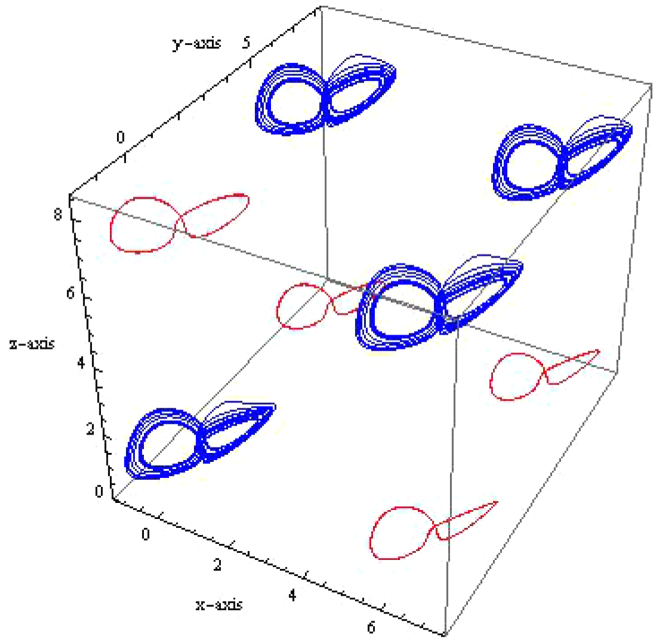

FIG. 1. The phase portrait of system (3.1) with $R_0 = 17$.

Figures 1–3 are the phase portraits of the system (3.1) with $R_0 = 17, 18$, and 19, respectively, where the initial conditions are taken as $(1 + 2\pi i, 1 + 2\pi j, 1 + 2\pi k)$, $i, j, k = 0, 1$.

Next, the numerical evidence for the sensitive dependence on initial conditions is presented, where one could take any two initial conditions very close to each other, to observe that their corresponding orbits will diverge to two different chaotic attractors. For example, take $R_0 = 20$. Figure 4 shows the initial conditions $(2.524, 1, 1)$ (in blue) and $(2.526, 1, 1)$ (in red). Figure 5 presents the phase portraits with the initial conditions $(2.5249, 1, 1)$ (in blue) and $(2.5251, 1, 1)$ (in red).

In what follows, an Rössler-type system is derived, which has infinitely many chaotic attractors.

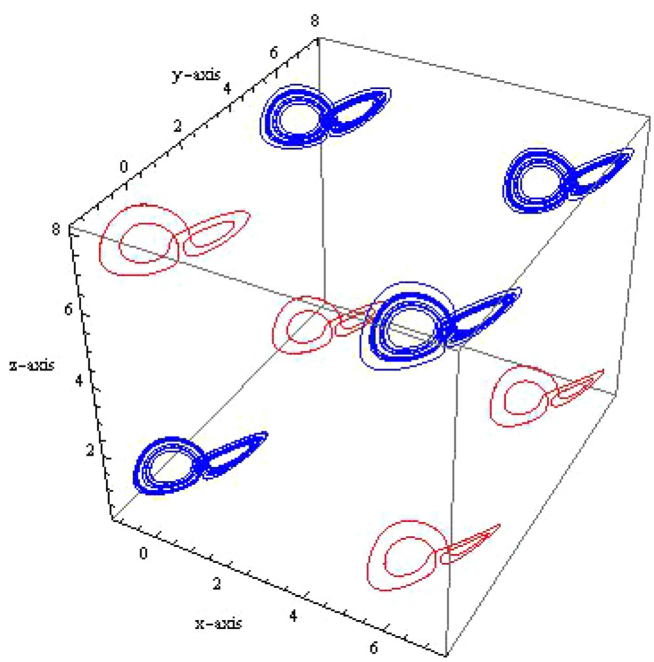

FIG. 2. The phase portrait of system (3.1) with $R_0 = 18$.

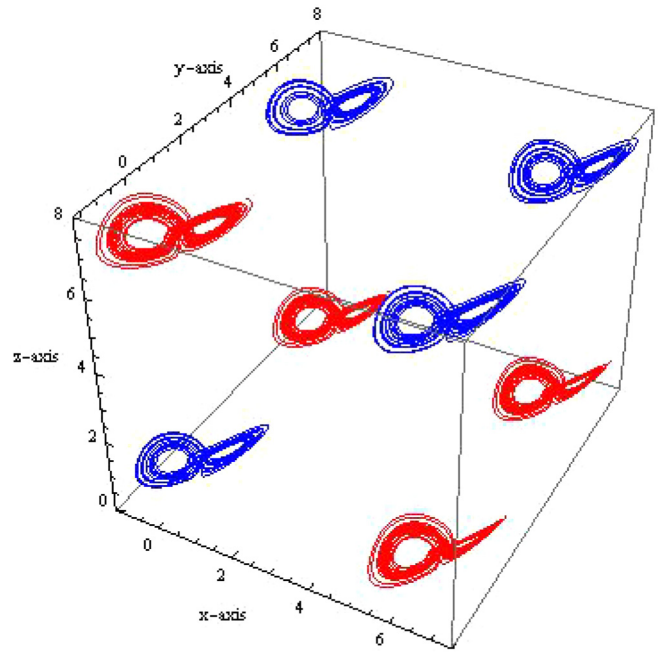

FIG. 3. The phase portrait of system (3.1) with $R_0 = 19$.

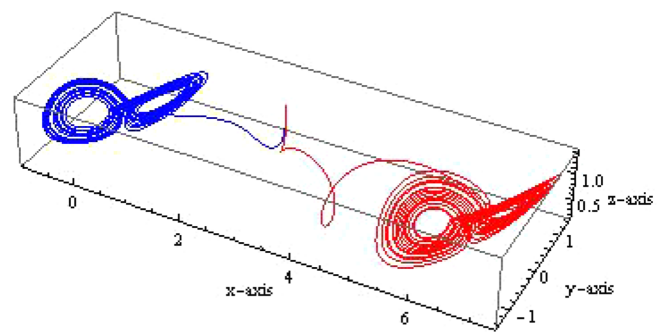

FIG. 4. The phase portraits of system (3.1) with $R_0 = 20$, and initial conditions $(2.524, 1, 1)$ (in blue) and $(2.526, 1, 1)$ (in red).

Start with the classical Rössler system, which has only a chaotic attractor

$$\begin{cases} \dot{p}_1 = -p_2 - p_3 \\ \dot{p}_2 = p_1 + 0.2p_2 \\ \dot{p}_3 = 0.2 + p_3(p_1 - 5.7). \end{cases}$$

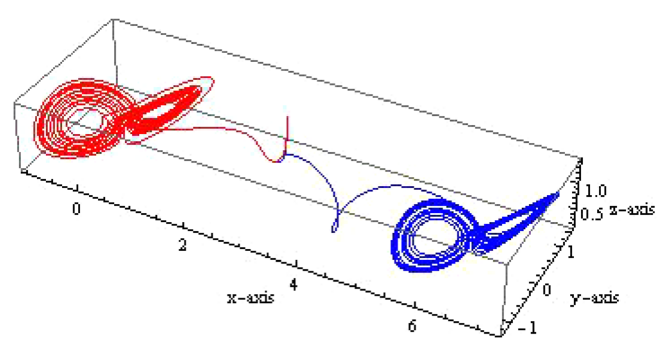

FIG. 5. The phase portraits of system (3.1) with $R_0 = 20$, and initial conditions $(2.5249, 1, 1)$ (in blue) and $(2.5251, 1, 1)$ (in red).

From the earlier approach, one can obtain the following Rössler-type system with infinitely many chaotic attractors:

$$\begin{cases} \dot{x}_1 = -\sin(x_2) - \sin(x_3) \\ \dot{x}_2 = \sin(x_1) + 0.2\sin(x_2) \\ \dot{x}_3 = \dfrac{0.2}{R_0} + \sin(x_3)\big(R_0\sin(x_1) - 5.7\big), \end{cases} \tag{3.2}$$

where R_0 is a positive constant.

Bifurcation analysis on the above system with bifurcating parameter R_0 has been performed, where R_0 only takes positive integers for simplicity. The numerical simulations show that the existence of periodic orbits is observable for $1 \le R_0 \le 12$; a long periodic orbit exists for $R_0 = 13$; the coexistence of many chaotic attractors is found for $R_0 \ge 14$.

Figures 6 and 7 are the phase portraits of the system (3.2) with $R_0 = 9$ and 15, respectively, and the initial conditions are taken as $(2\pi i, -0.0678 + 2\pi j, 0.02 + 2\pi k)$, $i, j, k = 0, 1$.

As an extension of the above methodology, consider the corresponding fractional-order systems.[5,6,9,22,27]

The Riemann-Liouville fractional integral is used, defined as follows [Ref. 19, Page 21]:

$${}_{t_0}D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} (t-s)^{\nu-1} f(s)\,ds, \tag{3.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function, $t_0 \in \mathbb{R}$ is the initial time, ν is a given positive number, and $\Gamma(\nu) = \int_0^\infty \tau^{\nu-1} e^{-\tau} d\tau$ is the Gamma function. For any $\beta \in (0, \infty)$, let m be a positive integer with $m - 1 < \beta \le m$. The fractional derivative in the sense of Caputo is[1]

$$D_C^{\beta} f(t) := {}_{t_0}D_t^{-(m-\beta)} \left(\frac{d^m f}{dt^m}(t) \right),$$

where β is the fractional order, and $f : \mathbb{R} \to \mathbb{R}$ is an mth differentiable function. Caputo fractional derivatives are popular in engineering applications and numerical computations.

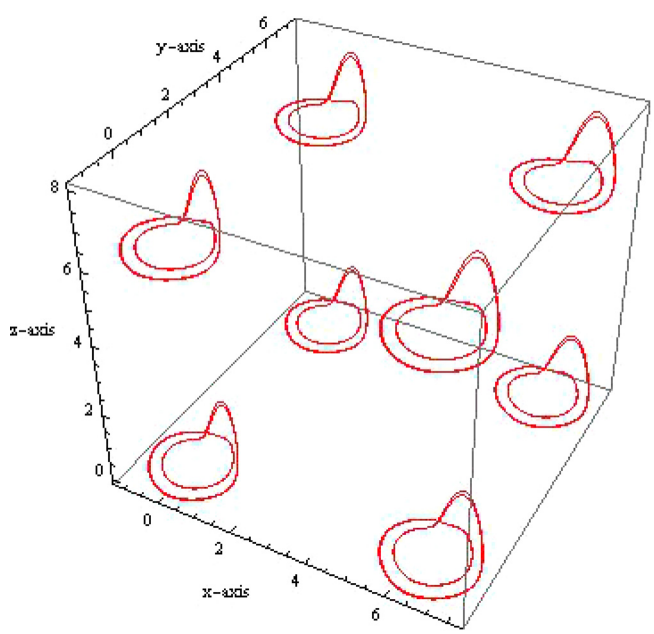

FIG. 6. The phase portrait of system (3.2) with $R_0 = 9$.

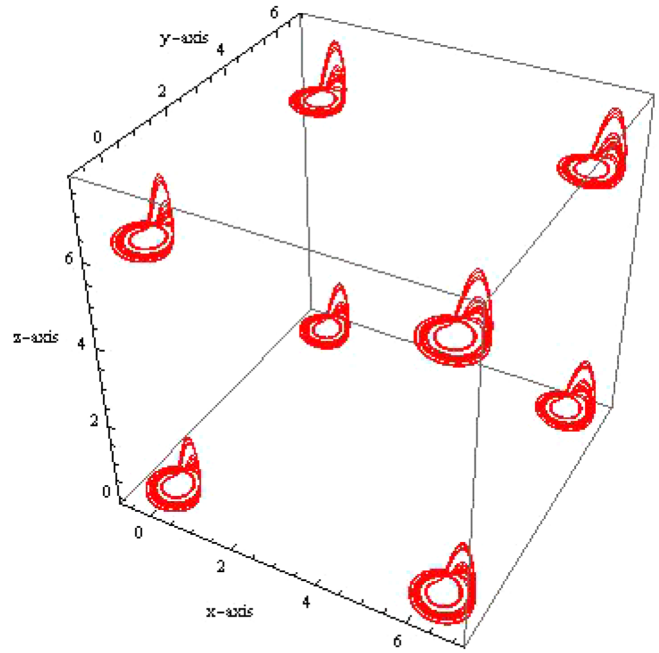

FIG. 7. The phase portrait of system (3.2) with $R_0 = 15$.

The fractional-order system corresponding to (3.1) is

$$\begin{cases} D_C^{\beta_1} x_1 = 10\big(\sin(x_2) - \sin(x_1)\big) \\ D_C^{\beta_2} x_2 = 28\sin(x_1) - 2R_0\sin(x_1)\sin(x_3) - \sin(x_2) \\ D_C^{\beta_3} x_3 = \dfrac{R_0}{2}\sin(x_1)\sin(x_2) - \dfrac{8}{3}\sin(x_3). \end{cases}$$

Finite difference schemes with non-uniform meshes were introduced to avoid the effect of the weakly singular kernel $(t-s)^{\nu-1}$ [$\nu \in (0,1)$ in (3.3)] in the fractional derivative and integral that affects the convergent rates of numerical computations.[15,20]

Here, the numerical algorithms developed in Ref. 15 are adopted. For $R_0 = 19$, and $\beta_i \approx 0.99$, $i = 1, 2, 3$, the existence of many chaotic attractors is observable, where the simulation diagrams are similar to those shown in Fig. 3. The 0-1 test is used to show that the fractional-order system has chaotic dynamics.[4]

IV. AN OPEN PROBLEM

Finally, inspired by the second part of Hilbert's sixteenth problem, an interesting problem is posed. In the method developed earlier, the functions ϕ_i could be some polynomials. In this setting, the number of the attractors for $\dot{\mathbf{x}} = \Phi\big(\phi_1(x_1), \phi_2(x_2), \phi_3(x_3)\big)$ can be controlled by the degrees of the polynomials ϕ_i, $1 \le i \le 3$, as shown by the following autonomous system:

$$\begin{cases} \dot{x}_1 = 10\big(x_2 - \phi_1(x_1)\big) \\ \dot{x}_2 = 28\phi_1(x_1) - 2R_0\phi_1(x_1)x_3 - x_2 \\ \dot{x}_3 = \dfrac{R_0}{2}\phi_1(x_1)x_2 - \dfrac{8}{3}x_3, \end{cases} \tag{4.1}$$

where $\phi_1(x_1) = a\,x_1(x_1 - 1)(x_1 - 2)$ and a is a parameter.

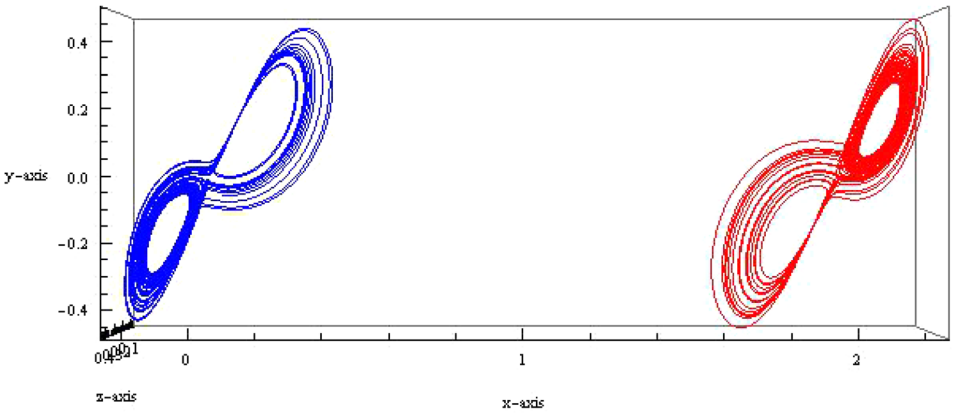

FIG. 8. The phase portrait of system (4.1) with $R_0 = 20$ and $a = 0.7$.

Figure 8 shows the numerical results, with $R_0 = 50$ and $a = 0.7$, where the orbit (in blue) starts from the initial conditions $(2.524, 1, 1)$, and the orbit (in red) starts from the initial conditions $(2.526, 1, 1)$. Two attractors are observable, where this number two is less than the degree of ϕ_1.

ACKNOWLEDGMENTS

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